Liquid Film Drain from an Accelerating Tank Wall

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Nomenclature

g = acceleration body force
t = time
u = the x component of velocity
x = distance measured along the tank wall from the position of the bulk liquid interface at $t = 0$
y = distance measured perpendicular to the tank wall
δ = thickness of draining film
μ = dynamic viscosity of liquid
ρ = density of liquid
ρ_v = density of vapor

IN many studies of a draining cryogenic tank, such as those of wall heat transfer or tank pressurization, it is important to have a knowledge of the liquid film that clings to the tank wall. Such a film is illustrated in Fig. 1. Several investigators have examined a draining film for various reasons in the past. This work has been both analytical and experimental and is described by van Rossum.[1] It is all for constant g.

Neglecting inertia terms in the momentum equation, the velocity in such a laminar draining film with no slip at the wall and zero shear at the liquid-vapor interface is given by[2]

$$u = \frac{(\rho - \rho_v)g}{\mu}\left(y\delta - \frac{y^2}{2}\right) \tag{1}$$

The continuity equation written for a slab of incompressible liquid dx in thickness is

$$\frac{\partial}{\partial x}\int_0^\delta u\,dy + \frac{\partial \delta}{\partial t} = 0 \tag{2}$$

One should note in passing that mass transfer from the liquid film introduces a nonzero term into the right-hand side of Eq. (2).

Substitution of the velocity distribution from Eq. (1) into the continuity equation yields

$$\frac{(\rho - \rho_v)g}{\mu}\,\delta^2\frac{\partial \delta}{\partial x} + \frac{\partial \delta}{\partial t} = 0 \tag{3}$$

Equation (3) is a quasi-linear, partial differential equation. The solution of such equations using the method of characteristics is described by Hildebrand.[3] In this case, application of the method reduces to the solution of the following two ordinary differential equations:

$$\frac{dx}{(\rho - \rho_v)g\delta^2/\mu} = \frac{dt}{1} = \frac{d\delta}{0} \tag{4}$$

The third term of this equation signifies that δ is a constant along a characteristic. This fact simplifies the integration of the equation formed by the first and second terms. This equation is

$$dx = \frac{(\rho - \rho_v)g\delta^2}{\mu}\,dt \tag{5}$$

Integration with the boundary condition $x = 0$ at $t = 0$ yields the desired film thickness

$$\delta = \left(\frac{\mu x}{(\rho - \rho_v)\int_0^t g\,dt}\right)^{1/2} \tag{6}$$

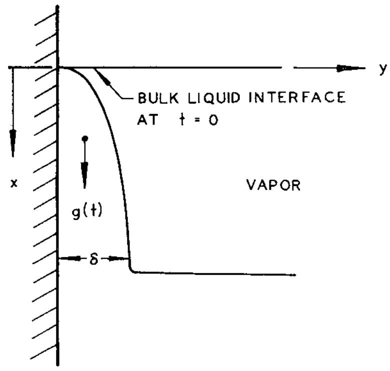

Fig. 1 Physical model.

For a constant g, Eq. (6) reduces to

$$\delta = \left(\frac{\mu x}{(\rho - \rho_v)g\,t}\right)^{1/2} \tag{7}$$

This agrees with the result of van Rossum except that he has neglected ρ_v in comparison to ρ.

Equation (6) is an interesting result since it reveals that the factor of importance in the film profile is the area under the g vs time curve. According to this analysis, which neglects surface tension and contact angle, the profile is unchanged during periods of zero g.

References

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Calculation of Natural Modes of Vibration for Free-Free Structures in Three-Dimensional Space

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HAVING had occasion to require the calculation of natural free-free modes for large three-dimensional structures, we have devised a procedure for developing the free-free matrix for a structure in three dimensions employing lumped mass and inertia concepts. The extension of this concept to three dimensions was indicated in Ref. 1.

Consider a three-dimensional structure consisting of lumped masses and inertias clamped at some point (the origin). Each of the lumped mass-inertia elements that constitute the structure possesses six degrees of freedom; three translational and three rotational. A matrix of flexibility influence coefficients $[C]$ exists which relates the six deflections and rotations of each lumped mass-inertia to forces and moments applied at every lumped mass of the structure. A right-hand coordinate system is employed.

The eigenvalue problem to be solved for the cantilevered modes of the structure can be written as

$$\{\delta\} = \omega^2[C]\lceil M \rfloor\{\delta\} \tag{1}$$

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In the foregoing equation, ω is the natural frequency; δ represents the deflections of the lumped masses of the structure u_i, v_i, w_i, θ_i, ϕ_i, ψ_i; and $\lceil M \rfloor$ is the mass matrix. The mass matrix is diagonal, and six terms along the diagonal are required to describe the mass properties of each mass-inertia element m_i, m_i, m_i, I_{xi}, I_{yi}, and I_{zi}; the last three terms representing the moments of inertia of each element (about axes through the element).

Let us consider one such element of the structure (m_1, I_{x1}, I_{y1}, I_{z1}). Equation (1) can be rewritten as

$$\begin{Bmatrix} u_1 \\ \theta_1 \\ v_1 \\ \phi_1 \\ w_1 \\ \psi_1 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \omega^2 [C] \begin{bmatrix} m_1 & & & & & & & \\ & I_{x1} & & & & & & \\ & & m_1 & & & & & \\ & & & I_{y1} & & & & \\ & & & & m_1 & & & \\ & & & & & I_{z1} & & \\ & & & & & & \cdot & \\ & & & & & & & \cdot \end{bmatrix} \begin{Bmatrix} u_1 \\ \theta_1 \\ v_1 \\ \phi_1 \\ w_1 \\ \psi_1 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \quad (1a)$$

Now release the holding point by allowing the origin the six degrees of freedom u_0, θ_0, v_0, ϕ_0, w_0, ψ_0 relative to a fixed external reference datum. The equivalent of Eq. (1a) is now

$$\begin{Bmatrix} u_1 \\ \theta_1 \\ v_1 \\ \phi_1 \\ w_1 \\ \psi_1 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} - \begin{bmatrix} 1 & 0 & 0 & z_1 & 0 & -y_1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -z_1 & 1 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & y_1 & 0 & -x_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} u_0 \\ \theta_0 \\ v_0 \\ \phi_0 \\ w_0 \\ \psi_0 \end{Bmatrix} = \omega^2 [C] \lceil M \rfloor \begin{Bmatrix} u_1 \\ \theta_1 \\ v_1 \\ \phi_1 \\ w_1 \\ \psi_1 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \quad (2)$$

where x_1, y_1, z_1 refer to the coordinates of this mass-inertia element. The dots indicate similar terms for the other elements that constitute the structure.

The force and moment equilibrium equations can be written as

$$\lfloor m \rfloor \{u\} = \lfloor m \rfloor \{v\} = \lfloor m \rfloor \{w\} = 0 \quad (3a)$$

$$[I_x]\{\theta\} + [y]\lceil m \rfloor\{w\} - [z]\lceil m \rfloor\{v\} = 0 \quad (3b)$$

$$[I_y]\{\phi\} + [z]\lceil m \rfloor\{u\} - [x]\lceil m \rfloor\{w\} = 0 \quad (3c)$$

$$[I_z]\{\psi\} + [x]\lceil m \rfloor\{v\} - [y]\lceil m \rfloor\{u\} = 0 \quad (3d)$$

where $\lceil m \rfloor$ represents the matrix formed by selecting only those elements of $\lceil M \rfloor$ which are associated with motion in but one direction.

Rewrite Eq. (2) as

$$\{\delta\} - [B]\{\delta_0\} = \omega^2 [C]\lceil M \rfloor\{\delta\} \quad (4)$$

Premultiply Eq. (4) by $\lceil M \rfloor$ and then by the transpose of $[B]$, yielding

$$[B]^T\lceil M \rfloor\{\delta\} - [B]^T\lceil M \rfloor[B]\{\delta_0\} = \omega^2[B]^T\lceil M \rfloor[C]\lceil M \rfloor\{\delta\} \quad (5)$$

Since $[B]^T\lceil M \rfloor\{\delta\}$ constitutes the left-hand side of Eqs. (3), it can be dropped from Eq. (5). Hence,

$$\{\delta_0\} = -\omega^2[G]^{-1}[B]^T\lceil M \rfloor[C]\lceil M \rfloor\{\delta\} \quad (6)$$

where

$$[C] = [B]^T\lceil M \rfloor[B] = \begin{bmatrix} M_T & 0 & 0 & S_z & 0 & -S_y \\ 0 & I_{xT} & -S_z & -P_{xy} & S_y & -P_{xz} \\ 0 & -S_z & M_T & 0 & 0 & S_x \\ S_z & -P_{xy} & 0 & I_{yT} & -S_x & -P_{yz} \\ 0 & S_y & 0 & -S_x & M_T & 0 \\ -S_y & -P_{xz} & S_x & -P_{yz} & 0 & I_{zT} \end{bmatrix} \quad (7)$$

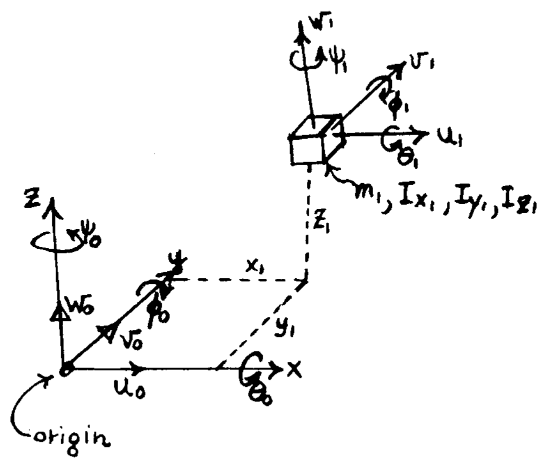

Fig. 1 Axis system of lumped mass-inertia element.

M_T represents the total structural mass; S_x, S_y, S_z represent the product of the total structural mass and the respective coordinates of the center of gravity; I_{xT}, I_{yT}, I_{zT} represent the total mass inertia about the three axes, and P_{xy}, P_{yz}, P_{xz} represent products of inertia about these axes.

Inserting Eq. (6) in Eq. (4) yields

$$\frac{1}{\omega^2}\{\delta\} = \left[\lceil I \rfloor - [B][G]^{-1}[B]^T\lceil M \rfloor\right][c]\lceil M \rfloor\{\delta\} \quad (8)$$

where $\lceil I \rfloor$ is the identity matrix. This now represents the eigenvalue system to be solved for "free-free" natural modes and frequencies.

The system of axes through the mass-inertia element of Fig. 1 is depicted as parallel to the axis system through the origin. In the analysis of a substructure, however, it may prove convenient to select an axis system for which this is not the case. There will exist a transformation matrix $[A]$ by which the deflections of the substructure can be related to the axes through the origin, i.e.,

$$\begin{Bmatrix} u \\ \theta \\ v \\ \phi \\ w \\ \psi \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = [A] \begin{Bmatrix} \bar{u} \\ \bar{\theta} \\ \bar{v} \\ \bar{\phi} \\ \bar{w} \\ \bar{\psi} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \quad (9)$$

where the barred symbols represent deflections relative to the substructure axes. When this consideration is incorporated in the foregoing development, the final eigenvalue system of equations will be

$$\frac{1}{\omega^2}\{\delta\} = \left[\lceil I \rfloor - [A][B][G]^{-1}[B]^T[A]^T\lceil M \rfloor\right][C]\lceil M \rfloor\{\delta\} \quad (10)$$

In Eq. (10), the deflections $\{\delta\}$ represent the deflections in terms of the axis convenient for each element.

This procedure has been programed for the IBM 7094 computing installation for as many as 600 total degrees of freedom. Where a substructure is relatively stiff in a particular direction, it may be desirable to define the substructure in terms of less elements for motion in this direction, and more lumped masses for motion in the other more significant directions. In this way, the structure can be represented three-dimensionally with a view toward economizing the number of mass-inertia elements employed.

References

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An Estimation Lemma for Laminar Compressible Boundary-Layer Velocity Profiles

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KNOWLEDGE of the velocity and temperature fields in a laminar steady two-dimensional boundary-layer requires the solution of a pair of coupled parabolic equations describing the flow of momentum and energy. It is much easier to obtain the same information for an incompressible boundary layer, since there the momentum equation is independent of the solution of the energy equation. The present note describes the use of the solution of a related incompressible problem to estimate the solution of the momentum equation of the original compressible problem. The fluid considered is an ideal gas, the viscosity of which is directly proportional to its temperature.

The compressible flow equations used here are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{\rho_1}{\rho}u_1u_1'(x) - \nu_1\frac{\rho_\infty}{\rho}\frac{\partial}{\partial y}\left(\frac{\rho_\infty}{\rho}\frac{\partial u}{\partial y}\right) = 0 \quad (1)$$

$$(\partial/\partial x)(\rho u) + (\partial/\partial y)(\rho v) = 0 \quad (2)$$

where

$$\nu_1 = (\mu_\infty/\rho_\infty)(P_1/P_\infty) \quad (3)$$

Here the subscript 1 denotes the limiting value of the dependent variable as y becomes very large, and the subscript ∞ denotes a reference value that will be determined later. The rest of the notation is conventional.

The set of equations would be completed by energy and state equations. Boundary conditions are specified at some upstream value of x, say at $x = 0$, at $y = 0$ (denoted by the subscript w), and as y becomes very large (denoted by the subscript 1). T_w, ρ_w, T_1, ρ_1, P_1, and u_1 are, in general, functions of x. The problem is now correctly posed for the four variables u, v, ρ, and T. The pressure $P(x, y)$ is assumed to be equal to the external flow value $P_1(x)$ for all y.

The energy equation is not used in the approximate method presented here. Instead, we assume only that the velocity and temperature increase monotonically with y from the wall values to the external flow values. The restriction to monotonic temperature profiles means that, for a given pressure field, Prandtl number, and ratio of wall temperature to external flow temperature, there is an upper bound on the external flow Mach number. For example, in the case of a flat plate in a gas of unit Prandtl number, the Crocco integral of the energy equation gives

$$M_1 \le \left[\frac{2}{\gamma - 1}\left(1 - \frac{T_w}{T_1}\right)\right]^{1/2}$$

for the temperature profile to be monotonic. Thus if $\gamma = 1.4$ and the wall temperature is $\frac{4}{5}$ of the external flow temperature, the Mach number can be no greater than 1, whereas if the wall temperature is $\frac{1}{5}$ of the external flow temperature, the Mach number can be as large as 2. For more general pressure fields and Prandtl numbers the assumption of monotonic temperature profiles will also imply an analogous limitation to moderate Mach numbers. Such a restriction to transonic or low supersonic Mach numbers would be necessary in any case if the linear viscosity-temperature law is expected to be a realistic approximation to the behavior of a diatomic gas. We also restrict our statements to cases where the external flow velocity is a nondecreasing function of x.

The problem is simplified by the introduction of the related independent variables x^* and y^*, which are given by the Howarth-Dorodnitsyn transformation

$$x^*(x) = x \quad (4)$$

$$y^*(x, y) = \frac{1}{\rho_\infty}\int_0^y \rho(x, t)dt \quad (5)$$

The transformed dependent variables will also be denoted by the superscript (*). For example,

$$u^*(x^*, y^*) = u(x, y) \quad (6)$$

Instead of Eqs. (1) and (2), we consider the pair of equations

$$u^*\frac{\partial u^*}{\partial x^*} + w^*\frac{\partial u^*}{\partial y^*} - \frac{\rho_1}{\rho^*}u_1u_1'(x) - \nu_1\frac{\partial^2 u^*}{\partial y^{*2}} = 0 \quad (7)$$

$$(\partial u^*/\partial x^*) + (\partial w^*/\partial y^*) = 0 \quad (8)$$

Note that w^* is merely a function defined by Eqs. (7) and (8); no statements are made about its relation to v or v^*.

In order to obtain the estimation lemma we will need

$$y^*(x, y) \le y \quad (9)$$

at fixed x and y. Examination of Eq. (5) shows that this is satisfied if

$$\rho/\rho_\infty \le 1 \quad (10)$$

everywhere in the boundary layer. This enables us to choose the reference values. We let T_∞ be the value of T_w at that x where P_1/T_w is greatest. Then P_∞ is P_1 at the same x, ρ_∞ is found from the ideal-gas law, and μ_∞ is found from a table of gas properties. This choice of T_∞ and P_∞ satisfies Eq. (10), which satisfies Eq. (9).

The earlier assumption that u is a monotonically increasing function of y implies that u^* is a monotonically increasing function of y^*, since

$$\frac{\partial u}{\partial y} = \frac{\partial y^*}{\partial y}\frac{\partial u^*}{\partial y^*} = \frac{\rho}{\rho_\infty}\frac{\partial u^*}{\partial y^*}$$

or

$$\frac{\partial u^*}{\partial y^*} = \frac{\rho_\infty}{\rho}\frac{\partial u}{\partial y}$$

and the right-hand side of this last equation is positive. Because of this result and Eq. (9), we see that Eq. (6) implies that

$$u(x, y) \le u^*[x^*(x), y^*(x, y)] \quad (11)$$

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